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A moving fluid interface on a rough surface

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When an interface between two fluids moves in contact with a solid boundary, the Navier–Stokes equations and the no-slip boundary condition provide an unsatisfactory theoretical model, because they predict an undefined velocity at the contact line and a non-integrable stress on the solid boundary. If the surface irregularities are included in the model, the flow on a length scale large compared with their size can be calculated, using a slip coefficient and treating the surface as smooth.

A simple type of corrugated surface is examined, and the effective slip coefficient calculated, for grooves of finite and infinite depth. The slip coefficient when the grooves are filled with one fluid and another fluid flows over them is also calculated. It is suggested that, when a fluid displaces another on a rough surface, the displaced fluid remains in the hollows on the surface, thus providing a partly fluid boundary for the displacing fluid and leading to a slip coefficient for the flow.

Fluid contained between two vertical plates and rising between them provides a simple example of a flow for which the solution can be found with and without a slip coefficient. With slip present, the force on the plates is finite and its value is calculated.

1. Introduction

Recent papers by Huh & Scriven (1971) and by Dussan V. & Davis (1974) have drawn attention to the difficulties encountered in attempting a theoretical description of a moving contact line between two fluids and a solid boundary. Such moving contact lines are of common occurrence, the most familiar being the water–air–glass contact line in a tilted tumbler. It is generally believed that the water, as it moves to wet a further portion of the glass surface, does so by rolling and not by sliding. The experiments described by Dussan V. & Davis show that particles of fluid initially on the fluid–fluid interface are found later to be on the solid–fluid interface and vice versa. Thus a condition of adherence, which as they explain is not the same as a no-slip boundary condition, is not maintained. Once a fluid particle has reached the solid surface it may not move along it and hence the no-slip condition can be satisfied. However, this description of the motion leads to two unsatisfactory conclusions: the fluid velocity is undefined at the contact line and the stress on the solid boundary has a non-integrable singularity, leading to an infinite force. These shortcomings of the model have often been encountered; in particular, by Huh & Scriven (1971), who calculate a variety of flows at a moving contact line, the interface between the two fluids remaining plane. The important contribution of Dussan V. & Davis was to show that the unsatisfactory nature of the model was not due to any approximations made in deducing particular flows, but was an inevitable consequence of using a continuum model of the flow together with the no-slip boundary condition.

There is, of course, no reason to suppose that a continuum model should give sensible results when distances of molecular dimensions are involved. A complete description of the motion at the contact line on a molecular scale would obviously be very difficult. Huh & Scriven suggest that, instead of examining this region in detail, its overall effect on the flow on a macroscopic length scale could be obtained by postulating a dynamic boundary condition, in which the relative velocity at a solid boundary is proportional to the velocity gradient there. The slip coefficient would be of the same order as the molecular length scale, but would have to be determined either experimentally or by consideration of the molecular processes involved at the contact line. Such a proposal would lead to a finite stress (or at least to an integrable one) and hence to a finite value for the force. Such a change in the usual no-slip boundary condition would in most circumstances be undetectable, since the additional term would only change the macroscopic flow by an amount proportional to the ratio of the molecular to the macroscopic length scales, except in special circumstances, of which the moving contact line is one. Then the stress would be inversely proportional to the molecular length scale near the contact line, leading to a force which could be large, but would be finite.

In the creeping-flow solutions without slip obtained by Huh & Scriven (1971), the solid boundary is assumed to be a geometrical plane. Irregularities of the surface on a molecular scale are one among many features which must be included in a full description of the flow near the contact line. Irregularities of solid surfaces, however, also occur on length scales much greater than the molecular scale. For example, a trace of a medium-ground steel surface reproduced by Hondros (1971) shows grooves about $2\,\mu$m deep with a lateral spacing of about $8\,\mu$m. The effect of such irregularities on the contact-line problem is dismissed by Huh & Scriven, who say that the singularity in the hydrodynamic model would still be present when it is applied to contact-line movement on scales less than the length scales of roughness and heterogeneity.

There are three length scales relevant to the problem: the macroscopic length scale, which may be the size of the fluid container and which is the appropriate length scale for the overall description of the flow; the microscopic scale, which measures the roughness of the surface; and the molecular length scale. The thesis of this paper is that, when the surface irregularities are included, a model, based on the Navier–Stokes equations, can be produced in which the difficulties hitherto encountered at the moving contact line are removed. The proposal is that (i) the contact line does *not* move relative to a solid boundary on the microscopic length scale and (ii) the *apparent* contact line, that is, the estimated position of the contact line on the macroscopic length scale, does move. The way in which it is suggested that the fluid moves is shown in figure 1, drawn on the microscopic length scale. Relative to the solid boundary S, fluid F_1 is moving to the right and is displacing fluid F_2. The initial contact line passes through C_0 and does not move. As F_1 advances, the F_1F_2 interface is drawn out, so that it becomes close to succes-

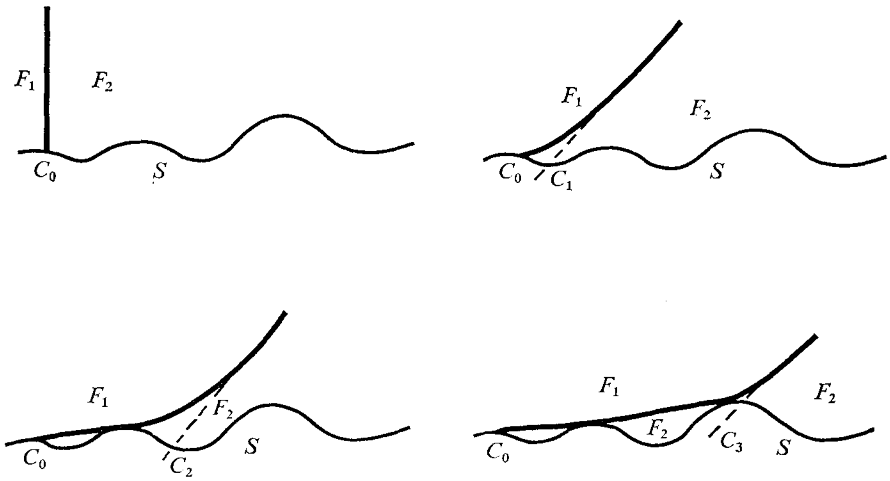

FIGURE 1. Sketch showing the suggested motion of the interface between two fluids F_1 and F_2 in contact with a solid boundary S. The motion is from left to right. C_0 is the initial position of the contact line, and C_1, C_2 and C_3 are subsequent positions of the apparent contact line, observed from a distance large compared with the scale of the irregularities of the surface.

sive peaks of the irregular solid surface. From a distance, the surface appears smooth and the apparent contact lines occupy the successive positions $C_1, C_2, \ldots$. Small amounts of F_2 are trapped in the crevices of the surface. The displacing fluid F_1 thus moves not over a solid surface, but over a surface partly or wholly composed of the displaced fluid F_2. The appropriate boundary condition to be applied on the macroscopic length scale is one which allows relative motion to occur between F_1 and S, because of the intervening fluid F_2, and the slip coefficient is of the same order as the spacing of the surface irregularities. There may, of course, be another molecular slip coefficient, to explain any motion of the initial contact line C_0. But the present proposal is that flow over rough surfaces can be adequately modelled by treating them as geometrically smooth and using a slip boundary condition. A somewhat similar proposal was made by Taylor (1971) with regard to the appropriate boundary condition for flow over a porous solid, although, in that case, there can be transport of fluid within the porous solid.

One feature of the proposed model perhaps needs further explanation. The diagrams in figure 1 show narrow regions near the peaks of the surface, where a thin film of F_2 separates F_1 from S. The draining of a thin film between two rigid surfaces is known to involve large normal forces which, on a macroscopic length scale, prevent contact in a finite time. However, in normal circumstances, the width of the gap is rapidly reduced to a molecular size. On the scale of the diagram, therefore, it is reasonable to suppose that F_1 makes contact with part of S, whether or not there is a monolayer of F_2 remaining between F_1 and S.

The simplified model of the surface used in the analysis presented in this paper consists of a uniform, periodic sequence of corrugations with straight crests and troughs, and the flow is in a direction perpendicular to the corrugations. Bowden & Tabor (1974) state that only in exceptional circumstances is the local slope of the surface inclined to the mean position of the surface at an angle greater than 5°. Although the corrugations are shallow, solutions are obtained for both shallow and very deep corrugations, since the latter extreme case is amenable to more exact mathematical analysis.

The rest of this paper contains an attempt to give quantitative support to the proposed model of the moving contact line. The flow of a fluid over a corrugated surface is first considered, the grooves having finite or infinite depths. Next, the flow over corrugations containing a different fluid is considered. In both cases an effective slip coefficient is obtained. The length scale is such that the Reynolds number based on the spacing of the grooves is small enough for the Stokes equations to be used. Finally, the motion of fluid rising between two vertical plates is considered, and the force on the plates is calculated.

2. Shear flow over a corrugated surface

The plane surface of a solid, on the microscopic scale, exhibits surface irregularities in the form of randomly distributed peaks and hollows. The very simplified model of a rough surface considered here consists of a regular succession of parallel grooves. In terms of Cartesian co-ordinates $\hat{x}$ and $\hat{y}$, parallel and normal to the mean position of the surface, the solid boundary has equation

$$\hat{y} = h\eta = hd\{-1+\cos(\hat{x}/h)\}, \tag{2.1}$$

where $2dh$ and $2\pi h$ are the depth and spacing of the grooves, respectively. When $d = 0$, the surface is plane and when $d = \infty$, the surface consists of a row of parallel plates at $\hat{x} = 2n\pi h$ (n integral) and $\hat{y} < 0$. Flow parallel to the crests of such a surface was considered by Taylor (1971) and Richardson (1971) in their model for flow over a porous surface.

On the macroscopic length scale, U and a are a typical velocity and a length, respectively. The flow near the surface is a uniform shear

$$u = U\hat{y}/a, \quad v = 0, \tag{2.2}$$

where u and v are the velocity components in the $\hat{x}$ and $\hat{y}$ directions, respectively. On the microscopic length scale, we can introduce non-dimensional co-ordinates and a stream function, defined by

$$\hat{x} = hx, \quad \hat{y} = hy, \quad u = (Uh/a)\,\partial\Psi/\partial y, \quad v = -(Uh/a)\,\partial\Psi/\partial x, \tag{2.3}$$

and the boundary conditions are

$$\Psi = \partial\Psi/\partial y = 0 \quad \text{on} \quad y = \eta, \tag{2.4}$$

$$\Psi \sim \tfrac{1}{2}y^2 \quad \text{as} \quad y \to \infty. \tag{2.5}$$

If we suppose that the Reynolds number for this inner flow,

$$Re = Uh^2/(a\nu), \tag{2.6}$$

is small enough for the Stokes equations to hold, the equation for Ψ is the biharmonic equation,

$$\{\partial^2/\partial x^2 + \partial^2/\partial y^2\}^2\,\Psi = 0. \tag{2.7}$$

The periodic nature of the boundary indicates a periodic stream function and the appropriate solution of (2.7) can be written in the form

$$\Psi = \tfrac{1}{2}y^2 + \beta y + \chi + \sum_{n=1}^{\infty} (b_n y + c_n)\, e^{-ny}\cos nx, \tag{2.8}$$

where b_n, c_n, β and χ are coefficients to be found. The value of β determines the effective slip coefficient, since the flow at large distances from the surface is

$$u \sim U(\hat{y}+\beta h)/a, \tag{2.9}$$

and this is the velocity which would be obtained for flow over the plane surface $y = 0$ if the boundary condition there were

$$\beta h\, \partial u/\partial \hat{y} = u. \tag{2.10}$$

Thus βh is the effective slip coefficient for the corrugated surface. The next step is to determine its value as a function of d.

If the boundary conditions (2.4) are applied to the stream function given by (2.8) and the resulting expressions expanded as Fourier series in x, two sets of equations for the determination of the unknown coefficients are obtained. The integrals occurring in the Fourier coefficients can be evaluated in terms of Bessel functions and the equations for b_n, c_n, β and χ are

$$\sum_{n=1}^{\infty}(-1)^n d e^{nd}\{-P_{n,m}+Q_{n,m}\}b_n+\sum_{n=1}^{\infty}(-1)^n e^{nd}P_{n,m}c_n \\ +(\tfrac{3}{4}d^2-d\beta+\chi)\,\delta_{0m}-\tfrac{1}{2}(d^2-d\beta)\,\delta_{1m}+\tfrac{1}{8}d^2\,\delta_{2m}=0, \tag{2.11}$$

$$\sum_{n=1}^{\infty}(-1)^n e^{nd}\{P_{n,m}+ndQ_{n,m}\}b_n-\sum_{n=1}^{\infty}(-1)^n n\, e^{nd}P_{n,m}c_n \\ +(\beta-d)\,\delta_{0m}+\tfrac{1}{2}d\delta_{1m}=0, \tag{2.12}$$

for $m = 0, 1, 2, \ldots$. In these equations, δ_{im} is the Kronecker delta and

$$P_{n,m}=\tfrac{1}{2}\{I_{n+m}(nd)+I_{n-m}(nd)\}, \tag{2.13}$$

$$Q_{n,m}=\tfrac{1}{2}(P_{n,m+1}+P_{n,m-1}), \tag{2.14}$$

where

$$I_k(z)\equiv\frac{1}{2\pi}\int_0^{2\pi}\exp(z\cos\theta)\cos k\theta\, d\theta, \tag{2.15}$$

a modified Bessel function.

For small values of d, these equations can be solved by truncation to give sufficiently accurate results. For larger values of d, however, the method fails to give meaningful results with double-precision arithmetic. The reason for the failure is that, when n is large, there is a very great difference between the values of the exponential terms in (2.8) at the crests and troughs of the surface. The matrix becomes ill conditioned and an accurate solution depends on computation to a very high degree of accuracy. The results obtained by the numerical solution are shown in table 1, for a series of values of d. The roughness factor R of the surface is also shown. This is defined as the ratio of the surface area to its projection on the plane $y = 0$. The numerical results suggest that β is approaching a limiting value β_∞ as $d \to \infty$, with $\beta_\infty - \beta \propto d^{-\frac{1}{2}}$. If this suggestion is true, and if β_1 and β_2 are successive values of β corresponding to a successive values d_1 and d_2 of d, the quantity

$$\beta^* = (\beta_2 d_2^{\frac{1}{2}} - \beta_1 d_1^{\frac{1}{2}})/(d_2^{\frac{1}{2}} - d_1^{\frac{1}{2}}) \tag{2.16}$$

should approach β_∞ as d increases. The values of β^* are given in the last column

d	R	β	β^*
0·2	1·01	0·161	—
0·4	1·04	0·256	0·49
0·6	1·09	0·310	0·54
0·8	1·12	0·342	0·55
1.0	1.22	0.364	0·55
1·2	1·30	0·380	0·55
1·4	1·40	0·39	0·55
1·6	1·52	0·40	0·55
1·8	1·60	0·41	0·56

TABLE 1. Single-fluid slip flow

of table 1, and the hypothesis is verified, with $\beta_\infty \simeq 0{\cdot}55$. For small values of d, the numerical results suggest that

$$\beta = d(1-d+0{\cdot}7d^3), \tag{2.17}$$

approximately. For values of d greater than 1·0, the slip coefficient is only weakly dependent on the depth of the grooves. This is because, when the grooves are sufficiently deep, the fluid in them is almost stagnant and increasing the depth further does not change the flow above the surface.

Since the calculation of β could be performed for only a limited range of values of d, and since the details of the flow could be accurately determined for only an even smaller range, an alternative method was used for the limiting case $d = \infty$, when the surface consists of a row of semi-infinite parallel plates. The periodic nature of the flow enables the solution to be found in a single strip. With the origin for x displaced to a position midway between two plates, the solution of the biharmonic equation is required in the strip $0 \leqslant x \leqslant \pi$, the conditions on the central plane being

$$\partial\Psi/\partial x = \partial^3\Psi/\partial x^3 = 0 \quad \text{on} \quad x = 0. \tag{2.18}$$

On the plate, the conditions are

$$\Psi = \partial\Psi/\partial x = 0 \quad \text{on} \quad x = \pi,\ y < 0, \tag{2.19}$$

and, because the stream function in $y > 0$ is an even function of $\pi - x$, we must have

$$\partial\Psi/\partial x = \partial^3\Psi/\partial x^3 = 0 \quad \text{on} \quad x = \pi,\ y > 0. \tag{2.20}$$

We also require that

$$\Psi \sim \tfrac{1}{2}y^2 \quad \text{as} \quad y \to \infty. \tag{2.21}$$

If we define $c(y)$ and $d(y)$ by

$$\Psi = c(y) \quad \text{on} \quad x = \pi,\ y > 0, \qquad \partial^3\Psi/\partial x^3 = d(y) \quad \text{on} \quad x = \pi,\ y < 0, \tag{2.22}$$

and take a Fourier transform in y, defined by

$$\overline{\Psi}(x,s) = \int_{-\infty}^{\infty} e^{isy}\,\Psi(x,y)\,dy, \tag{2.23}$$

the problem can be solved by the Wiener–Hopf technique. The appropriate form of $\bar{\Psi}$ can be written as

$$\bar{\Psi} = A(s)\cosh sx + B(s)\,x\sinh sx, \tag{2.24}$$

and, if

$$C_+(s) = \bar{\Psi}(\pi, s), \quad D_-(s) = \partial^3\bar{\Psi}(\pi, s)/\partial x^3, \tag{2.25}$$

application of the boundary conditions and elimination of A and B yield the equation

$$-\pi^{-1}s^2\sinh^2 s\pi\, C_+ = (4s\pi)^{-1}(\sinh 2s\pi + 2s\pi)\, D_-. \tag{2.26}$$

Let σ_m, $m = 1, 2, \ldots$, be the roots of

$$\sin 2\sigma\pi + 2\sigma\pi = 0 \tag{2.27}$$

in the first quadrant. Then we can write

$$(4s\pi)^{-1}(\sinh 2s\pi + 2s\pi) = K_+(s)\,K_-(s), \tag{2.28}$$

where

$$K_+(s) = e^{\alpha s}\prod_{m=1}^{\infty}\left[\left(1+\frac{s}{i\sigma_m}\right)\left(1+\frac{s}{i\tilde{\sigma}_m}\right)\exp\left\{-s\left(\frac{1}{i\sigma_m}+\frac{1}{i\tilde{\sigma}_m}\right)\right\}\right] \tag{2.29}$$

and

$$K_-(s) = K_+(-s). \tag{2.30}$$

The coefficient α is arbitrary and $\tilde{\sigma}_m$ is the complex conjugate of σ_m. The values of σ_m for large m are given by

$$\sigma_m = m - \tfrac{1}{4} - \frac{\ln(4m-1)\,\pi}{\pi^2(4m-1)} + \frac{i}{2\pi}\ln(4m-1)\,\pi, \tag{2.31}$$

and accurate values of the roots can be found by Newton iteration, with the values given by (2.31) as their initial estimates. The factorized form of (2.26) is

$$\frac{-\pi s^4 C_+(s)}{\{\Gamma(1-is)\}^2 K_+(s)} = \{\Gamma(1+is)\}^2 K_-(s)\,D_-(s) = M(s), \tag{2.32}$$

where $M(s)$ is an integral function, and where we have used

$$\sinh s\pi\,\Gamma(1+is)\,\Gamma(1-is) = s\pi. \tag{2.33}$$

To determine $M(s)$, the asymptotic form of $K_+(s)$ for large s is needed. We can make use of the asymptotic form of σ_m given by (2.31) and define

$$\begin{aligned} L_+(s) &= \prod_{m=1}^{\infty}\left[\left(1+\frac{s}{i(m-\frac{1}{4})}\right)^2\exp\left(\frac{-2s}{i(m-\frac{1}{4})}\right)\right] \\ &= \{\Gamma(\tfrac{3}{4})/\Gamma(\tfrac{3}{4}-is)\}^2\exp\{-2is\psi(\tfrac{3}{4})\} \end{aligned} \tag{2.34}$$

and

$$L_-(s) = L_+(-s). \tag{2.35}$$

It is then easy to show that, as $s\to\infty$,

$$K_+/L_+ \sim k\exp\{s(\alpha-\alpha')\}, \tag{2.36}$$

where k is a constant and α' is defined by

$$i\alpha' = \sum_{m=1}^{\infty}\left(\frac{1}{\sigma_m}+\frac{1}{\tilde{\sigma}_m}-\frac{2}{m-\frac{1}{4}}\right). \tag{2.37}$$

Hence, as $s \to \infty$ in the upper half-plane, the coefficient of $C_+(s)$ in (2.32) has the asymptotic value

$$(-is)^{-\frac{1}{2}} s^4 \exp\left[-s\{\alpha - \alpha' - 2i\psi(\tfrac{3}{4})\}\right]. \tag{2.38}$$

To obtain an algebraic behaviour at infinity, we choose

$$\alpha = \alpha' + 2i\psi(\tfrac{3}{4}). \tag{2.39}$$

If $c(y)$, the value of Ψ on $x = \pi$, is proportional to $y^{\frac{3}{2}}$ as $y \to 0$, $C_+ \sim s^{-\frac{5}{2}}$ and $M(s) = O(s)$. Also, as $s \to 0$, $C_+ \sim -is^{-3}$, and so

$$M(s) = i\pi s \tag{2.40}$$

is an acceptable solution. It is easily found that no other possibility is consistent with the stated conditions and keeps the velocity at the edge of the plate finite. Hence

$$C_+(s) = -is^{-3}\{\Gamma(1-is)\}^2 K_+(s) \tag{2.41}$$

and

$$c(y) \sim \tfrac{1}{2}y^2 + (i\alpha - 2\gamma)\, y \quad \text{as} \quad y \to \infty. \tag{2.42}$$

The slip coefficient is βh, as before, with

$$\beta = i\alpha - 2\gamma = -2\psi(\tfrac{3}{4}) - 2\gamma + \sum_{m=1}^{\infty}\left(\frac{1}{\sigma_m} + \frac{1}{\tilde{\sigma}_m} - \frac{2}{m-\frac{1}{4}}\right), \tag{2.43}$$

where γ is Euler's constant, and a numerical computation gives the value

$$\beta = 0{\cdot}5569, \tag{2.44}$$

which is in agreement with the asymptotic value deduced from the solutions for corrugations of finite depths

The functions A and B in the definition of the stream function (2.24) can now be found and the stream function calculated. In $y < 0$, a series of eddies is formed, of rapidly decreasing strength as y decreases. The flow between the plates away from the vicinity of the shear flow in $y > 0$ is similar to that obtained by Moffatt (1964) for the flow induced between two infinite plates by rotating a cylinder between them. Some streamlines in the region $y < 0$ are shown in figure 2. The most significant feature is the small distance the streamline through the edges of the plates penetrates into the region $y < 0$, the maximum depth of this streamline being only 6 % of the distance between the plates. The low value of the depression of this streamline explains why the slip coefficient is not very sensitive to changes in the depth of the grooves. The largest value of d for which numerical solutions were obtained corresponds to a depth/width ratio of only 0·57, so that even shallow grooves produce an effective slip coefficient when fluid flows over them.

The results of Richardson (1971) for flow parallel to the crests of a corrugated surface gave values of the slip coefficient comparable to those obtained here. For a row of plates, Richardson obtained $\beta = \frac{1}{2}\ln 2 = 0{\cdot}347$.

The results so far obtained are for a very simple type of rough surface. Generally, the surface will contain a random distribution of irregularities. For a surface consisting of shallow grooves, the results suggest that a slip coefficient proportional to some mean value d_m of the depths of the surface irregularities can

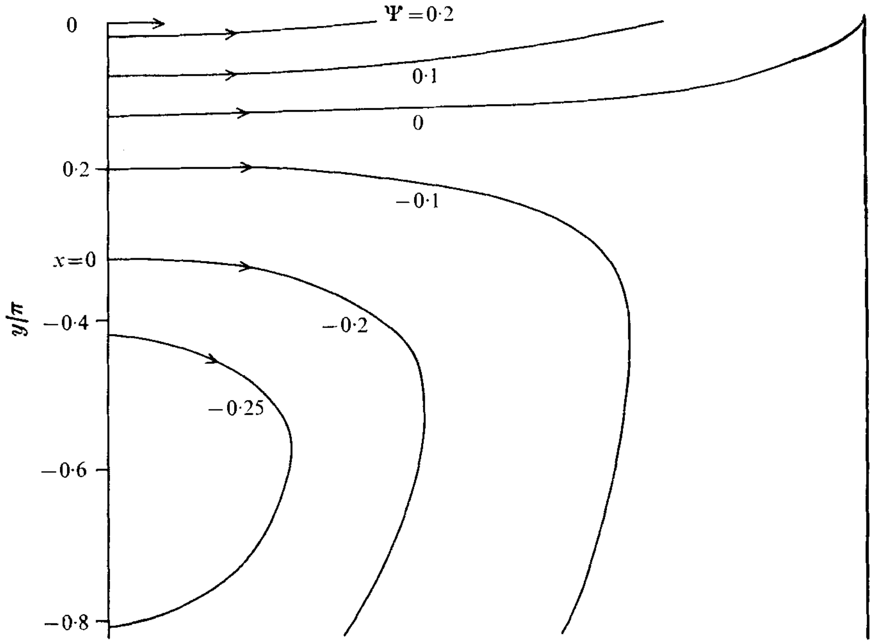

FIGURE 2. Streamlines of the flow between the sides of a very deep groove.

be defined. For deep grooves, on the other hand, the slip coefficient is proportional to some mean value h_m of the spacing of the grooves. The boundary condition to be applied on a macroscopic scale for flow over a rough surface which is moving with speed U in its own plane is

$$c_s\,\partial u/\partial y = u - U, \tag{2.45}$$

where y is measured from the surface into the fluid and where the slip coefficient c_s is given by

$$c_s = \tfrac{1}{2}d_m(1 - d/h) \tag{2.46}$$

for shallow grooves ($d/h < 0{\cdot}5$) and by

$$c_s = \beta h_m/2\pi \tag{2.47}$$

for deep grooves ($d/h > 1$), a suitable average value for β being about 0·4.

The shear flow past a wavy boundary has previously been calculated by Richardson (1973), in his discussion of the origin of the no-slip boundary condition. Since the use made of the calculation by Richardson is quite different from that of the present paper, it is worth noting the distinct aims of the two authors. Both show that, at large distances from the boundary, the velocity profile has the form

$$u = U(y + c_s)/a. \tag{2.48}$$

Richardson calculates c_s both when there is no slip at the boundary and when slip is allowed there. He deduces that, whatever boundary condition is applied on the microscopic scale, the macroscopic flow corresponds to the presence of a slip coefficient c_s which is negligible on the macroscopic scale. Hence, to leading order, there is no slip on the macroscopic scale. The argument presented here starts with the assumption of no slip on the microscopic scale. The effective slip coefficient which is then calculated enables an approximate calculation to be made for the flow above the crests of the corrugations on the boundary. The approximation consists of replacing the actual boundary by a fictional plane through the crests and applying a slip boundary condition there, with slip coefficient c_s and velocity profile (2.48). As in Richardson's treatment, there is no slip on the macroscopic scale, to leading order. It is not, of course, suggested that the approximation is formally correct, since the scale on which the slip coefficient is important is identical with the scale of the corrugations of the boundary. But *some* approximation must be made if the problem of the motion of two fluids past a rough boundary is to be made tractable and the small distortion of the streamlines above the crests of the corrugations indicates that the replacement of the real boundary by the fictional plane through the crests may be acceptable.

The method used by Richardson (1973) to find the flow over a wavy boundary was to map the region above one wavelength of the boundary onto the exterior of a circle. For the family of boundary shapes he uses, a closed-form solution for the slip coefficient is obtained. Provided the mapping is sufficiently simple, such closed-form values could be obtained for other boundary shapes. Although the complex-variable method of Richardson may be superior to that used in this section, it does not seem likely that it is suitable for the two-fluid problems of the following section, whereas the methods described here are readily adaptable.

A possible objection to the proposed fictional plane replacing the actual corrugated surface is that the slip coefficient depends on the position of the plane relative to the crests, and could even be negative. This would, of course, merely reflect the reverse flow present in the trough. If, however, the plane is taken at any level below the crests, the slip boundary condition would be appropriate only on the portions of the plane which are above the actual surface, and a different boundary condition would have to be used on the remainder to allow for the projection of the crests into the region above the plane. It is only when the plane passes through the crests that the actual flow and the fictional flow are in close approximation throughout the region above the plane. There is no attempt to give even an approximate description of the actual flow *below* the level of the crests.

3. Two-fluid flow

The previous section dealt with the situation to the left of the initial position C_0 of the contact line in figure 1, where a single fluid moves over a corrugated surface. To the right of this position, the displacing fluid is moving over a surface with its irregularities filled with the displaced fluid, which has been trapped in the hollows on the surface when the interface moves past them. Analysis of the motion near the moving interface is desirable but very difficult, and the only feasible case to consider seems to be the situation when the interface has moved some distance downstream, and the flow can again be regarded as steady. Even so, the two-fluid problem remains difficult, and we first consider the parallel-plate model for the

$\bar{\mu}$	β	$\bar{\kappa}_{max}$
0·01	0·0060	3·1
0·1	0·059	3·7
1·0	0·505	10·9
10	3·35	16·7
100	27·9	12·9

TABLE 2. Two-fluid flow: deep grooves

surface, which is the limiting case of very deep regular grooves. In addition, we suppose that the interface between the two fluids lies along the plane $y = 0$, which is the plane containing the edges of the plates. In the single-fluid solution for flow past such a surface, it was found that the streamline through the edges of the plates remained close to the plane $y = 0$. The proposed model for the two-fluid flow treats this streamline as the dividing streamline between the two fluids and forces it to lie exactly in the plane $y = 0$. This can only be done by the imposition of a normal stress across the interface. At a curved fluid interface, surface tension provides such a normal stress, and if it is sufficiently strong, the normal stress balance can be achieved with a negligibly small curvature of the surface. The surface tension at a fluid interface was also invoked by Huh & Scriven (1971) to justify their assumption that the fluid interfaces they were considering remained plane.

As in § 2, we can consider the flow in a single strip only, $0 \leqslant x \leqslant \pi$. We use suffixes 1 and 2 to denote quantities in the two regions $y > 0$ and $y < 0$, respectively, and μ_1 and μ_2 are the viscosities of the two fluids.

In the upper fluid, the stream function satisfies the conditions

$$\left.\begin{aligned} \partial\Psi_1/\partial x = \partial^3\Psi_1/\partial x^3 = 0 \quad &\text{on} \quad x = 0, \pi, \\ \Psi_1 = 0 \quad \text{on} \quad y = 0,& \\ \Psi_1 \sim \tfrac{1}{2}y^2 \quad \text{as} \quad y \to \infty,& \end{aligned}\right\} \tag{3.1}$$

and the appropriate solution of the biharmonic equation is

$$\Psi_1 = \tfrac{1}{2}y^2 + \beta y + \sum_{n=1}^{\infty} a_n\, y\, e^{-ny} \cos nx. \tag{3.2}$$

For $y < 0$, the boundary conditions are

$$\left.\begin{aligned} \partial\Psi_2/\partial x = \partial^3\Psi_2/\partial x^3 = 0 \quad &\text{on} \quad x = 0, \\ \Psi_2 = \partial\Psi_2/\partial x = 0 \quad &\text{on} \quad x = \pi, \\ \Psi_2 \to 0 \quad \text{as} \quad y \to -\infty,& \end{aligned}\right\} \tag{3.3}$$

and we can write the solution as

$$\Psi_2 = \sum_{m=1}^{\infty} e^{\sigma_m y}(x \sin \sigma_m x \cos \sigma_m \pi - \pi \sin \sigma_m \pi \cos \sigma_m x)\, b_m + \text{complex conjugate}, \tag{3.4}$$

where σ_m are the roots of (2.27).

The interfacial boundary conditions are continuity of tangential velocity and tangential stress at $y = 0$, and, in addition, we must have $\Psi_2 = 0$ there. When these conditions are applied, and when a Fourier analysis of Ψ_2 and its derivatives is made, the following sets of equations for the unknown coefficients a_n and b_m are obtained:

$$\sum_{m=1}^{\infty} \sigma_m^3(\sigma_m^2 - n^2)^{-2} b_m + \text{c.c.} = 0, \quad n \geqslant 0, \tag{3.5}$$

$$\sum_{m=1}^{\infty} \sigma_m^4(\sigma_m^2 - n^2)^{-2} b_m + \text{c.c.} = \begin{cases} \frac{1}{4}(-1)^{n+1} a_n, & n > 0, \\ -\frac{1}{2}\beta, & n = 0, \end{cases} \tag{3.6}$$

$$\sum_{m=1}^{\infty} \sigma_m^5(\sigma_m^2 - n^2)^{-2} b_m + \text{c.c.} = \begin{cases} (-1)^n \frac{1}{2}\bar{\mu} n a_n, & n > 0, \\ -\frac{1}{2}\bar{\mu}, & n = 0, \end{cases} \tag{3.7}$$

where $\bar{\mu} = \mu_1/\mu_2$ is the viscosity ratio. The coefficients can be determined by truncation of these infinite sets of equations and the values of β for various values of $\bar{\mu}$ are shown in table 2.

The value of β for $\bar{\mu} = 1$ should be compared with the value 0·56 obtained in § 2. The difference between these values is due to the displacement of the streamline through the plate edges, which is now forced to lie in the plane $y = 0$. Since the difference is only 10 %, the results obtained for this simple model may be used with some confidence, at least for fluids with comparable viscosity, even when surface tension is absent. An empirical formula constructed from the values of β given in table 2 is

$$\beta = 0{\cdot}6\bar{\mu}(1 + 0{\cdot}17\bar{\mu})/(1 + 0{\cdot}38\bar{\mu}). \tag{3.8}$$

The low values of β when the viscosity ratio is small are to be expected, since the more viscous lower fluid produces a large tangential stress on the upper fluid, reducing the amount of slip. These results suggest that, when a fluid of viscosity μ_1 is displacing a fluid of viscosity μ_2 over a rough surface, the appropriate slip coefficient is

$$c_{12} = \frac{1{\cdot}18\mu_1(\mu_2 + 0{\cdot}17\mu_1)}{\mu_2(\mu_2 + 0{\cdot}38\mu_1)} c_s, \tag{3.9}$$

where c_s was defined in (2.47) to be the slip coefficient when a fluid is moving over a deeply grooved surface whose crevices are clear of the presence of another fluid.

The pressure in the two fluids can be calculated from the values of Ψ_1 and Ψ_2. If p_1 and p_2 are the respective values of the pressure at the interface $y = 0$, we have

$$p_1 = (2\mu_1 U/a) \sum_{n=1}^{\infty} a_n n \sin nx, \tag{3.10}$$

$$p_2 = (2\mu_2 U/a)\left\{ \sum_{m=1}^{\infty} b_m \sigma_m \sin \sigma_m x \cos \sigma_m \pi + \text{c.c.} \right\}. \tag{3.11}$$

The curvature of the surface κ/h required to balance the normal stress difference across the interface is given by

$$T\kappa/h = p_1 - p_2 + 2\mu_2 \partial v_2/\partial y - 2\mu_1 \partial v_1/\partial y. \tag{3.12}$$

The proposed model, which has assumed that the surface remains plane, is selfconsistent provided the curvature calculated from (3.10)–(3.12) is negligibly small. The value of κ is

$$\kappa = (\mu_1+\mu_2)\,(Uh/Ta)\,\bar{\kappa}(x), \tag{3.13}$$

where
$$\bar{\kappa}(x) = 4(1+\bar{\mu})^{-1}\sum_{n=1}^{\infty}\sin nx \sum_{m=1}^{\infty}[(-1)^m\{\sigma_m^5\bar{\mu}^{-1}+n\sigma_m^2(\sigma_m^2-n^2)\} \times(\sigma_m^2-n^2)^{-2}b_m+\text{c.c.}]. \tag{3.14}$$

Values of $\bar{\kappa}$ were calculated at a number of values of x for each value of $\bar{\mu}$. The largest values $\bar{\kappa}_{\max}$ are given in table 2. Over the whole viscosity range tabulated, the model will be consistent provided that

$$(\mu_1+\mu_2)\,(Uh/Ta) \ll 10^{-1}. \tag{3.15}$$

This condition is not a stringent one. For example, if the two fluids are water and air, and we take $U = 10^{-2}\,\text{ms}^{-1}$, $a = 10^{-2}\,\text{m}$ and $h = 10^{-5}\,\text{m}$,

$$(\mu_1+\mu_2)\,(Uh/Ta) \approx 0{\cdot}3\times10^{-8}, \tag{3.16}$$

and the curvature of the surface produced by the unequal stresses on either side is minute.

We next consider the two-fluid flow over a surface with shallow grooves. In non-dimensional form, the equation of the surface is taken to be

$$y = d(-1+\cos x), \tag{3.17}$$

as in § 2, and we continue to assume that the fluid interface lies in the plane $y = 0$. The stream function for the flow in $y > 0$ is taken to be

$$\Psi_+ = \tfrac{1}{2}y^2+\beta y+\sum_{n=1}^{\infty}a_n y\,e^{-ny}\cos nx, \tag{3.18}$$

and in $y < 0$ we may write

$$\Psi_- = \tfrac{1}{6}\epsilon y^3+\tfrac{1}{2}\bar{\mu}y^2+\beta y+\sum_{n=1}^{\infty}(b_n y\,e^{-ny}+c_n y\,e^{ny}+d_n\sinh ny)\cos nx. \tag{3.19}$$

In choosing these expressions for the stream functions, the conditions

$$\Psi_+ = \Psi_- = 0 \quad \text{on} \quad y = 0 \tag{3.20}$$

have been satisfied, and so have the conditions of continuity of mean velocity and mean tangential stress at the interface. When these last two conditions are applied at each value of x, we obtain the equations

$$a_n = b_n+c_n+nd_n, \quad \bar{\mu}a_n = b_n-c_n, \qquad (3.21),\ (3.22)$$

so that the coefficients d_n can be eliminated, and Ψ_- can be written in terms of the two sets of coefficients b_n and c_n. These coefficients can be determined from the boundary conditions

$$\Psi_- = \partial\Psi_-/\partial y = 0 \tag{3.23}$$

on the boundary (3.17). As in § 2, after the boundary value of y has been substituted, a Fourier expansion yields two sets of equations for b_n and c_n, with coefficients which can be expressed in terms of Bessel functions.

$\bar{\mu}$	$d = 0{\cdot}1$	$d = 0{\cdot}2$
0·01	0·250	0·248
0·1	0·249	0·247
1·0	0·244	0·237
10	0·206	0·192
100	0·109	0·094

TABLE 3. Values of $\beta/\bar{\mu}d$ for shallow grooves

The numerical solution, even for small values of d, did not appear to converge as the number of equations was increased. The probable reason for this failure is the presence of cusps in the region occupied by the second fluid, where the boundary and the fluid interface meet. To avoid this difficulty, the solution was found when the position of the boundary was not given by (3.17) but by

$$y = d(-1-\delta+\cos x), \tag{3.24}$$

where δ took the values 0·4, 0·3, 0·2 and 0·1. The values of β were found for these values of δ and the value for $\delta = 0$ found by extrapolation. There was no difficulty in determining the solution for non-zero δ, the largest number of equations needed being 30. Except when $\bar{\mu}$ was large, the values of β were found to be almost linear functions of δ, so that the extrapolated values could be determined quite accurately.

Some results are given in table 3 and they can be summarized as follows. For shallow grooves with slopes of a few degrees, and with depths d_m, the slip coefficient is given by

$$c_s = \begin{cases} 0{\cdot}125\bar{\mu}d_m, & \bar{\mu} \leqslant 1 \\ 0{\cdot}1\bar{\mu}d_m, & \bar{\mu} = 10, \\ 0{\cdot}05\bar{\mu}d_m, & \bar{\mu} = 100, \end{cases} \tag{3.25}$$

where $\bar{\mu}$ is the ratio of the viscosity of the fluid above the surface to that of the fluid filling the grooves.

4. Fluid rising between two parallel plates

Now that some justification for the use of a slip boundary condition at a rough surface has been presented, it is possible to go on to examine the effect of applying such a condition to a flow involving a moving interface between two fluids in the presence of a solid boundary. Even for low Reynolds number flows and with the no-slip boundary condition, such two-fluid flows are difficult to describe theoretically. It is possible to find local similarity solutions near the contact line, if the fluid interface is assumed to remain plane. In terms of polar co-ordinates with origin at the contact line, two-dimensional flows have a stream function of the form

$$\Psi = rf(\theta), \tag{4.1}$$

and Huh & Scriven (1971) examine a variety of such flows. Earlier, Moffatt (1964) had considered a special case, when the region on one side of the fluid interface was void. As Huh & Scriven point out, a slip boundary condition prevents the use of similarity solutions of the form (4.1) in isolation, because there is a length scale present in the boundary condition. Solutions consisting of expansions in ascending or descending powers of r are possible, but a more satisfactory procedure is to use a Mellin transform of Ψ for motion with a plane solid boundary and a plane fluid interface. Such solutions would have to be matched to outer solutions, away from the contact line, to obtain a description of a complete problem.

There is one configuration in which it is possible to find the solution, both globally and in the vicinity of the contact line. Suppose fluid is contained between two vertical plates a distance $2a$ apart and is rising with speed U relative to the plates. If the top surface of the fluid is assumed to be plane, with a contact angle of $\frac{1}{2}\pi$ at the plates, and if the Reynolds number is small, the motion of the fluid can be calculated, provided the fluid above the interface is of sufficiently low viscosity for its dynamics to be neglected. The solution of this problem has been presented by Bhattacharji & Savic (1965) and by Bataille (1966). Their solutions include a non-integrable stress at the contact line, as do those presented by Huh & Scriven (1971). The contact angle must be equal to $\frac{1}{2}\pi$ for the solution to retain its simple form; there is no immediate extension to other contact angles.

The same configuration can be solved when the slip boundary condition is applied, and in that case there is a finite force on the vertical plates. If x and y are non-dimensional Cartesian co-ordinates, with x measured horizontally and y vertically downwards from a point in the free surface midway between the plates, the region occupied by the fluid is $|x| < 1$, $y > 0$. Relative to these co-ordinates, the plates have a velocity U in the $+y$ direction and the interface is at rest. The Reynolds number Re is $2Ua\rho_1/\mu_1$, where μ_1 is the viscosity and ρ_1 the density of the fluid, and if Re is sufficiently small, the stream function $Ua\Psi$ for the flow satisfies the biharmonic equation and the following conditions:

$$\left.\begin{aligned} \Psi = \partial^2\Psi/\partial y^2 = 0 \quad &\text{on} \quad y = 0, \\ \Psi = 0 \quad \text{on} \quad &x = \pm 1, \\ \partial\Psi/\partial x \pm \lambda\, \partial^2\Psi/\partial x^2 = -1 \quad &\text{on} \quad x = \pm 1. \end{aligned}\right\} \tag{4.2}$$

The last condition is the slip boundary condition (2.45), with $\lambda = c_{12}/a$, from (3.9) or (3.25).

The solution of these equations is

$$\Psi = -\frac{4}{\pi}\int_0^\infty \frac{\sin sy(x\cosh sx \sinh s - \sinh sx \cosh s)}{s(\sinh 2s - 2s + 4\lambda s \sinh^2 s)}\, ds. \tag{4.3}$$

As $y \to \infty$, the asymptotic value of the stream function is

$$\Psi \sim \tfrac{1}{2}(x - x^3)(1 + 3\lambda)^{-1}, \tag{4.4}$$

and the slip is unimportant away from the free surface, provided $\lambda \ll 1$. To examine the nature of the flow near the intersection of the free surface and the plates, it is convenient to change to new variables, defined by

$$x = -1 + \lambda\xi, \quad y = \lambda\eta, \quad s = \sigma/\lambda, \tag{4.5}$$

and the leading term in the stream function (4.3) is

$$\Psi = -\frac{2\lambda\xi}{\pi}\int_0^\infty \frac{e^{-\sigma\xi}\sin\sigma\eta}{\sigma(1+2\sigma)}d\sigma. \tag{4.6}$$

This integral can be expressed in terms of the exponential integral to give

$$\Psi = -2\lambda\xi\pi^{-1}[\tan^{-1}(\eta/\xi)+\mathrm{Im}\{\exp\{\tfrac{1}{2}(\xi+i\eta)\}E_1\{\tfrac{1}{2}(\xi+i\eta)\}\}]. \tag{4.7}$$

In terms of polar co-ordinates centred at the contact line and defined by

$$\xi = \rho\cos\phi, \quad \eta = \rho\sin\phi, \tag{4.8}$$

we have

$$\Psi = -\frac{2\lambda\rho\cos\phi}{\pi}\left[-\phi\sum_1^\infty\frac{(\frac{1}{2}\rho)^n\cos n\phi}{n!}+(-\gamma+\ln(2/\rho))\sum_1^\infty\frac{(\frac{1}{2}\rho)^n\sin n\phi}{n!}\right.$$
$$\left.+\sum_1^\infty\frac{(\frac{1}{2}\rho)^n\sin n\phi}{n!}\left(1+\tfrac{1}{2}+\ldots+\frac{1}{n}\right)\right], \tag{4.9}$$

and the asymptotic expansion for large ρ is

$$\Psi \sim -\frac{2\lambda\rho\cos\phi}{\pi}\left[\phi-\frac{\sin\phi}{\frac{1}{2}\rho}+\frac{1!\sin 2\phi}{(\frac{1}{2}\rho)^2}-\frac{2!\sin 3\phi}{(\frac{1}{2}\rho)^3}+\ldots\right]. \tag{4.10}$$

The leading term of this last result is the solution near the contact line when there is no slip. As $\rho\to 0$, the stream function with slip is shown by (4.9) to be $O(\rho^2\ln\rho^{-1})$ instead of $O(\rho)$, and it is this reduction in the size of Ψ near the contact line which makes the force on the plate finite.

The vertical component of the force on the plates can be calculated. The force per unit width on a length l of the plate $x = -1$ (measured from the free surface) is given by

$$F = \int_0^{l/a}\mu_1 U(\partial^2\Psi/\partial x^2)_{x=-1}dy$$
$$= \frac{8\mu_1 U}{\pi}\int_0^\infty\frac{\sinh^2 s\{1-\cos(sl/a)\}}{s(\sinh 2s-2s+4\lambda s\sinh^2 s)}ds. \tag{4.11}$$

For large values of the length of the plate relative to the gap between the plates, and for small values of the slip coefficient, a combination of asymptotic analysis and computation shows that (4.11) can be written as

$$F = \mu_1 U\left[\frac{3l}{a+3c_{12}}+\frac{4}{\pi}\ln(a/c_{12})-2{\cdot}178+O(a^2/l^2)+O(c_{12}/a)\right], \tag{4.12}$$

and it is clear that, to make this force finite, a non-zero value of the slip coefficient is required.

Surface tension must be present to ensure that the free surface remains essentially plane. The restriction on the size of the surface tension is more stringent than in the previous use of this condition, because of the increased length scale over which it is to hold, and we must have

$$(\mu_1+\mu_2)U/T \ll 1 \tag{4.13}$$

for the model to be valid.

5. Summary

It has been shown that, when an interface between two fluids moves in contact with a solid boundary, the continuum equations can be used to provide a consistent model of the flow, provided the surface irregularities are included. The effect of such irregularities is equivalent to the introduction of a slip coefficient, proportional to their depth if shallow and to their spacing if deep. On the basis of this model, a particular flow has been investigated and the force on the solid boundary has been shown to be finite, whereas the solution with no slip involves an infinite force.

The most serious gap in the argument is that it has not been possible to produce evidence in support of the picture, sketched in figure 1, of the moving position and shape of the interface. However, the most promising direction for further study would seem to be to produce estimates of the forces on solid boundaries when moving contact lines are present, as has been done for the especially simple case discussed in § 4. Comparison with experiment might then be able to decide between slip coefficients on the microscopic scale of the surface irregularities and on the molecular scale.

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